بعض خواص النظريات للتوزيعات الثنائية متسلسلة القوى

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ملخص البحث

يهدف هذاالبحث الى دراسة علاقات العزوم الأحادية والثنائية لتوزيعات متسلسلات القوى الثنائية المتغيرات ، والاستفادة من النتائج فى استخدام أمثلة جديدة لتوزيعات متسلسلات القوى الثنائية المتغيرات ، ويتناول هذا البحث تعريفا لتوزيعات متسلسلات القوى أحادية وثنائية المتغيرات . ويعرض البحث عدة أمثلة لمجموعة من التوزيعات الإحصائية المتقطعة الثنائية المتغيرات المشهورة ، وكذلك أمثلة لعدد من التوزيعات الجديدة ، وتمت دراسة التوزيعات الحدية والعزوم الأحادية والثنائية لتوزيعات متسلسلات القوى الثنائية المتغيرات ، ووجدنا عدة علاقات مميزة تتابعية للعزوم الأحادية وللعزوم الثنائية ، وتم استخدام هذه العلاقات لتحديد توزيعات متسلسلات القوى الثنائية المتغيرات بالضبط ، بالإضافة الى ذلك تم استخدام النتائج التى تم التوصل اليها فى استخدام أمثلة جديدة لتوزيعات متسلسلات القوى الثنائية المتغيرات .

١ ـ دكتور / رافد سعيد عبد الرزاق يعمل حاليا أستاذا مساعدا بكلية الادارة والاقتصاد جامعة قطر ، حاصل على بكالوريوس احصاء من الجامعة المستنصرية ببغداد عام ١٩٧٤ م ، وماجستير فى الإحصاء من جامعة بغداد عام ١٩٧٦ م ، وماجستير فى الرياضيات عام ١٩٨٢م بالإضافة الى ماجستير فى الاحصاء أيضا عام ١٩٨٢م ثم دكتوراه فى الاحصاء عام ١٩٨٣ م من جامعة ولاية بنسلفانيا الرسمية بأمريكا . تدرج فى عمله الأكاديمى من معيد حتى أستاذا مشاركا بجامعات بغداد منذ عام ١٩٧٥ م حتى ١٩٩١ م ، ثم عمل فى جامعة العلوم والتكنولوجيا بوهران من ١٩٩١-١٩٩٣ م ، ثم جامعة اليرموك من ١٩٩٣-١٩٩٤ م ، ثم جامعة آل البيت من ١٩٩٤- ١٩٩٦م ، وبعد ذلك جامعة قطر من عام ١٩٩٦ م حتى الآن . شغل عدة مناصب إدارية أكاديمية منها رئيسا لقسم الإحصاء جامعة بغداد من ١٩٩٨- ١٩٩١ م ، نشر مؤلفين و ٢٢ بحثا علميا .

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where $\lambda = de^{a+b+abc}$ and $\lambda_1 = a$, $\lambda_2 = b$, and $\lambda_{12} = abc$, See Patil and Joshi (1968), p. 67.

__Example 12__: Let $m(a,b,c) = a+abc$ and $m^*(a,b,c) = b+abc$. Let $M(a,b,c) = a+abc$. Also

$$\begin{aligned} D_b \ln Q(b,c) &= (b+abc) / b - ac \\ &= 1 + ac - ac = 1 \neq 0 \end{aligned}$$

It follows that $\ln Q(b,c) = b$, or $Q(b,c) = e^b$.

Hence $f(a,b,c) = e^{b\ u+abc} = e^{a+b+abc}$.

Therefore the bivariate rvc (X,Y) has jpmf

$$P(X=x,\ Y=y) = a^x\, b^y\, e^{-(a+b+abc)} \sum_{i=0}^{m} ci / [(x-i)!\ (y-i)\ i!]$$

$$x,y = 0,1,2,\ldots$$

and is 0 otherwise. where $m = \min(x,y)$. This defines the bivariate Poisson rvc with parameters λ_1, λ_2, and λ_{12} , where $\lambda_1 = a$, $\lambda_2 = b$ and $\lambda_{12} = abc$, see Ahmad (1961), p. 810.

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at (0,0), see Ahmad (1961).

In the following examples, all the conditions of Theorem 9 can be easily verified. Therefore, we give only the function M(a,b,c) and the jpmf.

Example 10: Let $m(a,b,c,n) = n.a/(1+a+b+c)$ and $m^*(a,b,c,n) = n.b/(1+a+b+c)$, where $0 \leq a, b, c, a+b+c < 1$ and $n = 1, 2, 3, \ldots$. Then let $M(a,b,c,n) = n.\ln(1+a+b+c)$ and the bivariate rvc (X,Y) ~ BPSD with jpmf

$$P(X=x,Y=y) = \frac{n!}{x!\, y!\, (n-x-y)!} \, \frac{(1+c)^{n-x-y}}{(1+a+b+c)^n} \, a^x b^x \qquad \begin{array}{l} x,y=0,1,2,\ldots,n \\ x+y \leq n, \end{array}$$

and is 0 otherwise. This defines a new bivariate rvc. Clearly the bivariate binomial rvc is a special case of this when c = 0. See Mardia (1970), p. 82.

Example 11: Let $m(a,b,c,d) = d(a+abc)e^{a+b+abc}$ and $m^*(a,b,c,d) = d(b+abc)\ e^{a+b+abc}$, where $0 \leq a, b, c, d < \infty$. Let $M(a,b,c,d) = e^{a+b+abc}$ and the bivariate rvc (X,Y) ~ BPSD with jpmf

$$P(X=x,Y=y) = a^x b^y \exp\{-d\, e^{a+b+abc}\} \sum_{n=0}^{\infty} d^n/n! \sum_{i=0}^{m} c^i n^{x+y-i}/[(x-i)!\,(y-i)!\,i!] \qquad x,y=0,1,2,\ldots,$$

and is 0 otherwise, where m = min(x,y).

This defines a new bivariate rvc. Clearly the bivariate Neyman's type A distribution (i) with parameters λ, λ_1, λ_2 and λ_{12} is a special case of this

It is interesting to note that in many cases of interest, D_b [ln Q(b,c)] = 0 implying that for example the mean of X is sufficient to determine the corresponding defining function f(a,b,c) as will be observed in the following examples.

Example 9: Let $m(a,b,c) = (a+abc)e^{a+b+abc}/(e^{a+b+abc} - 1)$ and $m^*(a,b,c) = (b+abc).e^{a+b+abc} /(e^{a+b+abc} - 1)$ be the means of the a marginal rvs X and Y respectively, where $0 \leq a, b, c \leq ?$. Let $M(a,b,c) = \ln(e^{a+b+abc} - 1)$. Then $D_aM(a,b,c) = (1+bc)e^{a+b+abc}/(e^{a+b+abc} - 1) = m(a,b,c)/a$.

Also $D_b \ln Q(b,c) = m^*(a,b,c)/b - D_bM(a,b,c) = 0$. Hence $Q^*(b,c) = 1$.

Therefore,

$$e^{M(a,b,c)} = e^{a+b+abc} - 1$$

$$= \sum_{\substack{x=0 \\ x+y\geq 1}}^{\infty} \sum_{y=0}^{\infty} \sum_{i=0}^{m} c^i a^x b^y / [(x-i)!\,(y-i)!\,i!]$$

where m = min (x,y). That is, $e^{M(a,b,c)}$ has the Maclaurin expansion in a and b with non-negative coefficient. Hence, all the conditions of theorem 9 are satisfied and therefore there exists a bivariate rvc (X,Y) taking values in $\{x, y = 0, 1, 2, \ldots; x+y \geq 1\}$ such that the mean of the marginal rv X is the given m(a,b,c). Also, the defining function of the bivariate rvc (X,Y) is $f(a,b,c) = e^{M(a,b,c)} = e^{a+b+abc} - 1$. Therefore,

$$P(X=x, Y=y) = \frac{a^x b^y}{e^{a+b+abc} - 1} \sum_{i=0}^{m} c^i/[(x-i)!\,(y-i)!\,i! \qquad x,y = 0, 1, 2, \ldots \quad x+y \geq 1,$$

and is 0 otherwise. This defines the bivariate Pisson rvc (X,Y) truncated

Let $0 \leq u < a < t < r_1$. Then for $(a,b,\underline{c}) \in [u,t]\times(0,r_2)\times\Lambda$, we have that

$$\int_u^t [\, m(a,b,\underline{c}) / a \,]\, d_a = \int_u^t D_a [\, \ln f(a,b,\underline{c}) \,]\, d_a$$

implies in $(a,b,\underline{c}) \in [u,t]\times(0,r_2)\times\Lambda$ that

$$f(t,b,\underline{c}) = F(u,b,\underline{c})\, e^{M(t,b,\underline{c})},$$

where $F(u,b,\underline{c}) = f(u,b,\underline{c})\, e^{-M(u,b,\underline{c})}$.

Changing t into a, we get $\forall(a,b,\underline{c}) \in [u,t]\times(0,r_2)\times\Lambda$ that

$$f(a,b,\underline{c}) = F(u,b,\underline{c}).\, e^{M(a,b,\underline{c})}$$

Again we can show following the argument given in the proof of Theorem 5 that

$$f(a,b,\underline{c}) = e^{M(a,b,\underline{c})} \qquad \forall(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda.$$

Since $f(a,b,\underline{c})$ has the Maclaurin expansion in a and b with non-negative coefficients, it follows that $e^{M(a,b,\underline{c})}$ has the Maclaurin expansion with the same property. Therefore (ii) is true. This completes the proof of the theorem.

We remark that Khatri (1959), p. 489, has stated that: given the means vector and the variance-covariance matrix of the bivariate rvc (X,Y) ~ BPSD it determines the defining function uniquely and in any other case, it is not determined uniquely. The same statement has been repeated by Patil (1960), p. 227. But it follows from our Theorem 9 that to determine the BPSD uniquely, it is enough to know the means of the marginal rvs of the bivariate rvc (X,Y).

and is 0 otherwise.

By construction the bivariate rvc (X,Y) ~ BPSD. Furthermore, the mean of the marginal rv X is

$$\mu'_{1,0} = E(X) = aD_a [\ \ln f(a,b,\underline{c})] \quad \forall(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda\backslash(0,0).$$

Using (11) we have that

$$\mu'_{1,0} = aD_a [\ \ln e^{M(a,b,\underline{c})}\ ]$$

$$= aD_a\, M(a,b,\underline{c})$$

Then $\forall(a,b,\underline{c}) \in (0,r_1)\times(0,r_2)\times\Lambda$ we have that

$$D_a\, M(a,b,\underline{c}) = \mu'_{1,0}\,/a$$

$$= m(a,b,\underline{c})/a.$$

Similarly we can show that

$$D_b\, M(a,b,\underline{c}) = \mu'_{0,1}\,/\,b$$

$$= m^*(a,b,\underline{c}).$$

(B): Let the means of the marginal rvs and the defining function of the bivariate rvc (X,Y) ~ BPSD be $m(a,b,\underline{c})$, $m^*(a,b,\underline{c})$ and $f(a,b,\underline{c})$ respectively. Then, from (8) in $(a,b,c) \in (0,r_1)\times(0,r_2)\times\Lambda$, we have

$$m(a,b,\underline{c})\,/\,a = D_a [\ \ln f\,(a,b,\underline{c})]$$

Let $M(a,b,\underline{c}) = \ln f(a,b,\underline{c})$. Then

$$D_a\, M(a,b,\underline{c}) = D_a [\ \ln f(a,b,\underline{c})]$$

$$= m(a,b,\underline{c})\,/\,a \qquad \forall(a,b,\underline{c}) \in (0,r_1)\times(0,r_2)\times\Lambda.$$

Similarly.

$$D_b\, M(a,b,\underline{c}) = D_b [\ \ln f(a,b,\underline{c})]$$

$$= m^*(a,b,\underline{c})\,/b \qquad \forall(a,b,\underline{c}) \in (0,r_1)\times(0,r_2)\times\Lambda.$$

Hence (i) is true.

Now, we ask: What functions $m(a,b,\underline{c})$ and $m^*(a,b,\underline{c})$ can be used as the marginal means of the bivariate rvc $(X,Y) \sim$ BPSD? The answer is given in

Theorem 9: Let $m(a,b,\underline{c})$ and $m^*(a,b,\underline{c})$ be well-defined, non-negative and continuos functions of $(a,b,\underline{c})$ $\forall(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda\backslash(0,0)$, for some r_1, r_2, and Λ. Then there exists a bivariate rvc $(X,Y) \sim$ BPSD and having $m(a,b,\underline{c})$ and $m^*(a,b,\underline{c})$ as the means of the marginal r.vs. X and Y respectively, if and only if

(i) there exists a function $M(a,b,\underline{c})$ such that $\forall(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda$,

$$D_a\, M(a,b,\underline{c}) = m(a,b,\underline{c})\,/a$$

and

$$D_b M\,(a,b,\underline{c}) = m^*(a,b,\underline{c})\,/\,b,$$

(ii) $e^{M(a,b,\underline{c})}$ has a Maclaurin expansion in a and b $\forall(a,b,\underline{c}) \in$ $[0,r_1)\times[0,r_2)\times\Lambda$ with non-negative coefficients.

Proof: **(A):** Using (i) we obtain $M(a,b,\underline{c})$, and then using (ii) we can write, for some $N \subseteq \{0, 1, 2, \ldots\}$, that

$$e^{M(a,b,\underline{c})} = \sum_{x,y\in N} K_{x,y}\,(\underline{c})\, a^x\, b^y \qquad \forall(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda.$$

Let

$$f(a,b,\underline{c}) = e^{M(a,b,\underline{c})}. \tag{11}$$

Then, we can define a bivariate rvc (X, Y) taking the value (x,y) with probability

$$\begin{aligned} P(X=x,\ Y=y) &= K_{x,y}(\underline{c})\, a^x\, b^y\, /\, e^{M(a,b,\underline{c})} \\ &= K_{x,y}\,(\underline{c}\,)\, a^x\, b^y\, /\, f\,(a,b,\underline{c}) \qquad x,y \in N \end{aligned}$$

Therefore, we have that

$$f_1(a,b,\underline{c}) = H(\underline{c})\, f_2(a,b,\underline{c}) \qquad \forall(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda.$$

If we consider $P(X=x,Y=y)$ first using $f_2(a,b,\underline{c})$ and then $f_1(a,b,\underline{c})$ it follows that the function $H(\underline{c})$ is arbitrary and then therefore can be take to be 1. Hence, we have

$$f_1(a,b,\underline{c}) = f_2(a,b,\underline{c}) \qquad \forall(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda.$$

Therefore, it follows that the bivariate rvcs (X_1,Y_1) and (X_2,Y_2) are equal. This completes the proof of the theorem.

An important conclusion can be obtained by using Theorems 3 and 5, which is given in

__Thereon 6__: The bivariate rvc $(X,Y) \sim$ BPSD is uniquely determined from its bivariate moments $\mu'_{r,s}$, $\mu'_{r+1,s}$ and $\mu'_{r,s+1}$.

Also, using Theorems 4 and 5, we obtain an important conclusion given in

__Theorem 7__: The bivariate rvc $(X,Y) \sim$ BPSD is uniquely determined from any two successive marginal moments of X and any two successive marginal moments of Y.

It is known, Al-Jasim (1975), Theorem 7, p. 39, that the rv $X \sim$ PSD is uniquely determined from its mean. This leads with the conclusion of Theorem 4^* to the following

__Theorem 8:__ The r.v $X \sim$ PSD is uniquely determined from any of its two successive moments.

$$\int_u^t [m(a,b,\underline{c}) / a]\, da = \int_u^t D_a [\text{In } f_1(a,b,\underline{c})]\, da$$

$$= \int_u^t D_a [\text{In } f_2(a,b,\underline{c})]\, da$$

which implies that

$$\text{In } f_1(t,b,\underline{c}) - \text{In } f_1(u,b,\underline{c}) = f_2(t,b,\underline{c}) - f_2(u,b,\underline{c}).$$

Therefore,

$$f_1(t,b,\underline{c}) = G(u,b,\underline{c})\, f_2(t,b,\underline{c}) \qquad \text{in } (a,b,\underline{c}) \in [u,t)\times(0,r_2)\times\Lambda,$$

where $G(u,b,\underline{c}) = f_1(u,b,\underline{c}) / f_2(a,b,\underline{c})$.

Changing t into a, we get $\forall(a,b,\underline{c}) \in [u,r_1)\times(0,r_2)\times\Lambda$ that

$$f_1(a,b,\underline{c}) = G\,(u,b,\underline{c})\, f_2\,(a,b,\underline{c}).$$

It is easily seen that $G(u,b,\underline{c})$ does not depend on u. Therefore, we can write $G(u,b,\underline{c}) = G(b,\underline{c})$. Hence, we have

$$f_1\,(a,b,\underline{c}) = G(b,\underline{c})\, f2\,(a,b,\underline{c}) \qquad \forall(a,b,\underline{c}) \in [u,r_1)\times(0,r_2)\times\Lambda.$$

Finally, we conclude that

$$f_1\,(a,b,\underline{c}) = G(b,\underline{c}).\, f_2\,(a,b,\underline{c}) \qquad \forall(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda.$$

Similarly by using the mean $m^*(a,b,\underline{c})$ we obtain that,

$$f_1\,(a,b,\underline{c}) = G^*(a,\underline{c})\, f_2(a,b,\underline{c}) \qquad \forall(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda.$$

Therefore, it follows that

$$f_1\,(a,b,\underline{c}) = G\,(b,\underline{c}).\, f_2\,(a,b,\underline{c})$$
$$= G^*(a,\underline{c}) \times f_2\,(a,b,\underline{c}).$$

Hence,

$$G(b,\underline{c}) = G^*(a,\underline{c}) = H(\underline{c}\,).$$

Therefore,

$$m^*(a,b,\underline{c}) / b = D_b [\ln f(a,b,\underline{c})]$$
$$= D_b [\ln Q(b,\underline{c})\, e^{M(a,b,\underline{c})}]$$
$$= D_b \ln Q(b,\underline{c}) + D_b M(a,b,\underline{c})$$

Hence,

$$D_b \ln Q(b,\underline{c}) = m^*(a,b,\underline{c}) / b - D_b M(a,b,\underline{c})$$

But the left hand side is independent of a, therefore the right hand side is also independent of a. We conclude,

$$D_b \ln Q(b,\underline{c}) = R(b,\underline{c}) \quad \text{say.}$$

Solving this simple differential equation gives

$$Q(b,\underline{c}) = Q^*(b,\underline{c})$$

Where without loss of generality, the arbitrary constant, which is in general a function of c, can be 1.

Finally, after substituting we obtain

$$f(a,b,\underline{c}) = Q^*(b,\underline{c})\, e^{M(a,b,\underline{c})} \qquad \forall (a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda.$$

Therefore, we can obtain the defining function of the bivariate rvc (X, Y) if the means of the marginal rvs X and Y are given.

To prove the uniqueness, suppose that (X_1, Y_1) and (X_2, Y_2) are two bivariate rvc having BPSD and that $\mu'_{1,0} = m(a,b,\underline{c})$ and $\mu'_{0,1} = m^*(a,b,\underline{c})$ for both. Assume that the corresponding defining functions are $f_1(a,b,\underline{c})$ and $f_2(a,b,\underline{c})$ respectively. Therefore for $(a,b,\underline{c}) \in (0,r_1)\times(0,r_2)\times\Lambda$ we have

$$m(a,b,\underline{c}) / a = D_a [\ln f_1(a,b,\underline{c})]$$
$$= D_a [\ln f_2(a,b,\underline{c})]$$

let $0 < u \le a \le t < r_1$. Then in $(a,b,\underline{c}) \in [u,t]\times(0,r_2)\times\Lambda$, we have

$$m(a,b,\underline{c})\,/a = D_a\,[\text{In } f(a,b,\underline{c})]$$

Therefore, for $(a,b,\underline{c}) \in [u,t]\times(0,r_2)\times\Lambda$ we have that

$$\int_u^t [\,m(a,b,\underline{c})\,/\,a]\,da \;=\; \int_u^t D_a\,[\text{In } f(a,b,\underline{c})]\,da$$

i.e.

$$M(t,b,\underline{c}) - M(u,b,\underline{c}) = \text{In } [f(t,b,\underline{c})\,/\,f(u,b,\underline{c})]$$

Where $M(a,b,\underline{c})$ is such that $D_aM(a,b,\underline{c}) = m(a,b,\underline{c})/a \;\forall\; (a,b,\underline{c}) \in [u,t]\times(0,r_2)\times\Lambda$. Therefore

$$f(t,b,\underline{c}) = Q(u,b,\underline{c})\; e^{M(t,b,\underline{c})}$$

where $Q(u,b,\underline{c}) = f(u,b,\underline{c})\; e^{M(t,b,\underline{c})}$. Changing t into a, we get $\forall(a,b,\underline{c}) \in [u,t]\times(0,r_2)\times\Lambda$ that

$$f\,(a,b,\underline{c}) = Q\,(u,b,\underline{c})\; e^{M(t,b,\underline{c})}. \tag{10}$$

In (10) the left hand side does not depend on u, but the first term of the right hand side depends on u. This is not possible. Therefore, $Q(u,b,\underline{c})$ does not depend on u and therefore it equal to $Q(b,\underline{c})$. Then (10) becomes

$$f(a,b,\underline{c}) = Q\,(b,\underline{c})\; e^{M(a,b,\underline{c})} \qquad \forall(a,b,\underline{c}) \in [u,r_1)\times(0,r_2)\times\Lambda.$$

Finally, we conclude that

$$f(a,b,\underline{c}) = Q\,(b,\underline{c})\; e^{M(a,b,\underline{c})} \qquad \forall(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda.$$

It will be seen little later that as far as the marginal rv X is concerned the function $Q(b,\underline{c})$ is arbitrary and therefore without loss of generality can be taken to be the constant 1. But $f(a,b,\underline{c})$ must satisfy another condition about the mean of the marginal rv Y. Assume that the mean of Y is $m^*(a,b,\underline{c})$. Then from (9) for $\forall(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda\backslash(0,0)$, we have

$$m^*(a,b,\underline{c}) = bD_b\,[\text{In } f(a,b,\underline{c})]$$

Theorem 4: The mean of the marginal rv X (or Y) of the bivariate rvc (X, Y) ~ BPSD is uniquely determined from any of its two successive marginal moments.

But the moments $\mu'_{r,0}$ of the bivariate rvc (X,Y) is the same as the moment μ'_r for the marginal rv X. Also, we have proved in Theorem 2 that in case that the rvc (X,Y) ~ BPSD, then the rv X ~ PSD. Therefore, we can think about X as any rv ~ PSD and hence the conclusion given in Theorem 4 can be restated as

Theorem 4*: The mean of the rv X ~ PSD is uniquely determined from any of its two successive moments.

6 Means of the Marginal RandomVariables.

The means of the marginal rvs X and Y which are $\mu'_{r,0}$ and $\mu'_{r,0}$, respectively, are well-defined, non-negative and continuous functions $\forall$ $(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda\backslash(0,0)$. Using (1) we can write

$$E(X) = \mu'_{1,0} = aD_a[\ln f(a,b,\underline{c})] \tag{8}$$

$$E(Y) = \mu'_{0,1} = bD_b[\ln f(a,b,\underline{c})] \tag{9}$$

We give now an important conclusion in this paper.

Theorem 5: The bivariate rvc (X,Y) ~ BPSD is uniquely determined from the means of the marginal rvs of the bivariate rvc (X,Y).

Proof: Suppose that the mean of the marginal rv X is $m(a,b,\underline{c})$. Then from (8) for $(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda\backslash(0,0)$ we have

$$m(a,b,\underline{c}) = aD_a[\ln f(a,b,\underline{c})]$$

Let $0 < u \leq a \leq t < r_1$. Then for $(a,b,\underline{c}) \in [u,t]\times(0,r_2)\times\Lambda$, we have that

This reduces to

$$aD_a\,\mu'_{r,s} = \mu'_{r+1,s} - \mu'_{r,s}\cdot\mu'_{1,0} \tag{2}$$

Similarly, we have

$$bD_b\,\mu'_{r,s} = \mu'_{r,s+1} - \mu'_{r,s}\cdot\mu'_{0,1} \tag{3}$$

To obtain the recurrence relation for the moments of the marginal rv X, which is also has a PSD in one active parameter a, let s = 0 in (2). A special case of this relationship was derived by Noack (1950), p. 128.

Rearranging the terms in (2) and (3) we get $\forall(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda\backslash(0,0)$, and $\forall$ r, s = 0 ,1, 2, ... that

$$\mu'_{1,0} = \mu'_{r+1,s}/\mu'_{r,s} - aD_a\,(\ln\mu'_{r,s}) \tag{4}$$

$$\mu'_{0,1} = \mu'_{r,s+1}/\mu'_{r,s} - bD_b\,(\ln\mu'_{r,s}) \tag{5}$$

Also, let s = 0 in (4) and let r = 0 in (5) to obtain that

$$\mu'_{1,0} = \mu'_{r+1,0}/\mu'_{r,0} - aD_a\,(\ln\mu'_{r,0}) \tag{6}$$

$$\mu'_{0,1} = \mu'_{0,s+1}/\mu'_{0,s} - bD_b\,(\ln\mu'_{0,s}) \tag{7}$$

We remark that properties (4) and (5) hold for all rvs having BPSD and for all r, s = 0 , 1, 2, ..., and properties (6) and (7) hold for the marginal moments.

We believe that these relations are not only new, but also contain important conclusions, which are give in

Theorem 3: Given the bivariate moments $\mu'_{r,s}$ and $\mu'_{r+1,s}$ (or $\mu'_{r,s}$ and $\mu'_{r,s+1}$) for the bivariate rvc (X,Y) having BPSD, then the mean of the marginal rv X (or Y) is uniquely determined.

f(a,b,$\underline{c}$) as its defining function. Also, the marginal rv Y ~ PSD and has the same function f(a,b,$\underline{c}$) as its defining function.

It is very important to note that the defining function for the bivariate rvc (X,Y) and the marginal r.vs. X and Y are the same function f(a,b,$\underline{c}$). To obtain the jpmf of the bivariate rvc (X,Y), we expand the function f(a,b,$\underline{c}$) as a power series in the two active parameters a and b. But to obtain the pmf of the marginal rv, say X, we expand the function f(a,b,$\underline{c}$) as a power series in only one active parameter a. This simple but important point was overlooked by the previous research workers in this subject.

5. Bivariate Moments

We define for all r, s = 0, 1, 2, ...,

$$\mu'_{r,s} = \mu'_{r,s}(a,b,\underline{c}) = E(X^r Y^s).$$

It is easily seen that

$$\mu'_{r,s} = [1/f(a,b,\underline{c})]\,(aD_a)^r\,(bD_b)^s\,f(a,b,\underline{c}) \qquad (1)$$

where D_a denotes the partial derivative with respect to a, and that the function $\mu'_{r,s}$ is a well-defined, non-negative, and continuous function $\forall\,(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda\backslash(0,0)$, for all r, s = 0, 1, 2, ...

Differentiating (1) with respect to a in $(a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda\backslash(0,0)$, we get

$$D_a\,\mu'_{r,s} = [1/f(a,b,\underline{c})]\,D_a(aD_a)^r\,(bD_b)^s\,f(a,b,\underline{c}) + [1/f(a,b,\underline{c})]^2 \cdot [-D_a f(a,b,\underline{c})]\,[aD_a)^r\,(bD_b)^s\,f(a,b,\underline{c}]$$

Example 8: Let $f(a,b,c,n) = (a + b + c)^n$, where $0 < a, b, c, a+b+c < 1$ and $n = 1, 2, \ldots$. Then

$$P(X=x, Y=y) = \frac{n!\ c^{n-x-y}}{x!\ y!\ (n-x-y)!} \frac{a^x\ b^y}{(a+b+c)^n} \qquad \begin{array}{l} x, y = 0,1,2,\ldots,n \\ x+y \leq n. \end{array}$$

This defines a new bivarate rvc. Clearly the bivariate binomial distribution is a special case of this when c approaches 1, see Mardia (1970), p. 82.

4. Marginal Distributions

Let the rvc $(X,Y) \sim$ BPSD. Let $P_X(x)$ be a function such that

$$P_X(x) = \sum_{y=0}^{\infty} P(X=x, Y=y)$$

$$= \sum_{y=0}^{\infty} K_{x,y}(\underline{c})\, a^x b^y / f(a,b,\underline{c})$$

$$= [1/f(a,b,\underline{c})]\, a^x \sum_{y=0}^{\infty} K_{x,y}(\underline{c})\, b^y$$

$$= B_x(b,\underline{c})\, a^x / f(a,b,\underline{c})$$

It is easy to see that $P_X(x)$ is a well-defined function $\forall (a,b,\underline{c}) \in [0,r_1)\times[0,r_2)\times\Lambda\backslash(0,0)$, $P_X(x) \geq 0\ \forall\ x$, and $\sum_{x=0}^{\infty} P_X(x) = 1$. Hence it follows that $P_X(x)$ is the pmf of the marginal rv X. Therefore we have.

Theorem 2: If the bivariate rvc $(X,Y) \sim$ BPSD and has the defining function $f(a,b,\underline{c})$, then the marginal rv $\sim$ PSD and has the same function

$$P(X=x, Y=y) = \frac{x^y}{x!y!} \frac{a^x b^y}{\text{Cosh}(ae^b)-1} \qquad x = 2, 4, 6, \ldots \quad y = 0, 1, 2, \ldots .$$

This defines a new biraariate rvc with parameters a and b.

Example 5: Let $f(a,b) = -1n(1-a-b)$, where $0 < a, b, a+b < 1$. Then

$$P(X=x, Y=y) = \frac{\Gamma(x+y)}{x!\, y!} \frac{a^x b^y}{-1n(1-a-b)^k} \qquad x, y = 0, 1, 2, \ldots \quad x+y \geq 1.$$

This defines the bivariate logarithmic rvc with parameters a and b. See Mardia (1970), p. 84.

Example 6: Let $f(a,b,c,n) = [-1n(1-ab-ac]^n$, where $0 < a, b, c, b+c < 1$ and $n = 1, 2, \ldots$. Then

$$P(X=x, Y=y) = S(n,x) \frac{nx!\, c^{x-y}}{y!\,(x-y)!} \frac{a^x\, b^y}{[-1n(1-ab-ac]^n} \qquad x = n, n+1, \ldots \quad y = 0, 1, 2, \ldots \quad y \leq x.$$

This defines a new bivariate rvc with parameters a, b, c, and n, where S(n,x) is the Stirling number of the first kind with arguments n and x.

Example 7: In Example 2, let $a = p_1/(1- p_1 - p_2)$ and $b = p_2/(1 - p_1 - p_2)$. Then $p_1 = a/(1 + a + b)$ and $p_2 = b/(1 + a + b)$. Therefore $0 \leq p_1, p_2, p_1+p_2 < 1$. Substituting these values in Example 2, we obtain that

$$P(X=x, Y=y) = \frac{n!}{x!\, y!\, (n-x-y)!} p_1^x\, p_2^y\, (1-p_1-p_2)^{n-x-y} \qquad x,y=0,1,2,\ldots,n \quad x+y \leq n$$

This defines the 2-dimensional (variate) multinomial rvc. with parameters n, p_1, and p_2, see Patil ad Joshi (1968), p. 68.

Example1: Let $f(a,b,c) = e^{a+b+abc}$, where $0 \le a, b, c < \infty$. Then

$$f(a,b,c) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \sum_{i=0}^{m} c^i a^x b^y / [(x-i)!\ (y-i)!\ i!],$$

where m = min (x,y). Therefore,

$$P(X=x, Y=y) = e^{-(a+b+abc)} a^x b^y \sum_{i=0}^{m} c^i / [(x-i)!\ (y-i)!\ i!] \qquad x, y = 0, 1, 2, \ldots,$$

and is 0 otherwise. This defines the bivariate Poisson rvc with parameters λ_1, λ_2, and λ_{12}, where $\lambda_1 = a$, $\lambda_2 = b$ and $\lambda_{12} = abc$, see Ahmad (1961), p. 810.

Example 2: Let $f(a,b,n) = (1+a+b)^n$, where $0 \le a, b, a+b < 1$ and n= 1,2,.... Then

$$P(X=x, Y=y) = \frac{n!}{x!y!(n-x-y)!} \frac{a^X b^Y}{(1+a+b)^n} \qquad x, y = 0, 1, 2, \ldots, n \quad x+y \le n$$

This defines the bivariate binomial rvc with parameters a, b and n, see Mardia (1970), p. 82.

Example 3: Let $f(a,b,k) = (1-a-b)^{-k}$, where $0 \le a, b, a+b < 1$ and $k > 0$. Then

$$P(X=x, Y=y) = \frac{\Gamma(k+x+y)}{\Gamma(k)\, x!y!} a^x b^y (1-a-b)^k \qquad x, y = 0, 1, 2, \ldots$$

This defines the negative binomial rvc with parameters k, a and b, see Mardia (1970), p. 84.

Example 4: Let $f(a,b) = \mathrm{Cosh}(ae^b) - 1$, where $0 < a, b < \infty$. Then

The Poisson, the binomial, the negative binomial, and the logarithmic series distributions are well-known members of the PSDs with defining functions e^a, $(1-a)^n$, $(1-a)^{-k}$, and $-\ln(1-a)$ respectively. See Abdulrazak and Patil (1986, 1994, 1996), and Patil et al (1988) for further details.

It is easy to see that the theorems proved in the special case of PSDs in one parameter are also true (with slight modification in the statement and the proof of the theorems) in the general case of s+1 parameters.

3. The Bivariate Power Series Distribution

To generalize the above definition to the case of bivariate power serioes distribution (BPSD), consider the function $f(a,b,\underline{c})$ where

$$f(a,b,\underline{c}) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} K_{x,y}(\underline{c})\, a_x\, b_y, \qquad K_{x,y}(\underline{c}) \geq 0$$

$\forall x,y$ and $\forall \underline{c} \in \Lambda \subseteq \Re^s$, and the double series converges $\forall (a,b,\underline{c}) \in [0,r_1) \times [0,r_2) \times \Lambda$. The bivariate random vector (rvc) (X,Y) having joint probability mass function (jpmf)

$$P(X=x, Y=y) = K_{x,y}(\underline{c})\, a^x\, b^y / f(a,b,\underline{c}) \qquad \forall\ x,y \geq 0$$

is said to have a BPSDs in (s+2) parameters. The parameters a and b are called the 'active parameters", $\underline{c}$ are called the "inactive parameters", and $f(a,b,\underline{c})$ will be called the "defining function".

We give below some examples of BPSDs.

stributions. Finally, Borzadaran, 2001, characterized some univariate discrete distributions including some PSDs via maximization of the entropy under certain constraints.

This excited our interest in the extension to the case of multivariate PSDs. In particular we wanted to know if the above remark by Patil (1966) is true or not. Also in the literature we found very few examples of the multivariate PSDs, so we wanted to construct more examples. To avoid the cumbersome notations we decided to deal only with the bivariate PSDs. The method of proof etc. can be easily generalized to the multivariate PSDs.

2. Definition of the Power Series Distributions

Generalizing the definition given by Patil (1962), we consider the function $f(a,\underline{c})$, Where $\underline{c} = (c_1,\ldots,c_s) \in \Lambda \subseteq \Re^s$, and

$$f(a,\underline{c}) = \sum_{x=0}^{\infty} K_x(\underline{c})\, a^x, \qquad K_x(\underline{c}) \geq 0$$

for all $x \geq 0$, and the series converges for $(a,\underline{c}) \in [0,r) \times \Lambda$, where r is the radius of convergence of $f(a,\underline{c})$. The random variable (rv) X having probability mass function (pmf) given by

$$P(X = x) = K_x(\underline{c})\, a^x / f(a,\underline{c}) \qquad \forall\ x \geq 0,$$

is said to have a PSD in s+1 parameters. We will abbreviated that by writing $X \sim$ PSD. We shall call the parameter a the "active parameter", the parameters $\underline{c}$ the "inactive parameters", and $f(a,b,\underline{c})$ will be called the "defining function".

Kosambi. Later Noack (1950) not knowing the work of Kosambi also studied the PSDs. But it was Khatri who gave a detailed exposition of the subject in 1959. In particular, he proved that the PSD is uniquely determined from its first two cumulates (moments) and that any other two successive cumulates cannot determine it uniquely. Furthermore, he considered the multivariate extension of the PSDs, and proved that all the means, variances and covariances together determine the multivariate PSDs uniquely. He illustrated his theorem by considering the multinomial and the negative multinomial distributions.

In a series of papers started in 1962, Patil has studied the univariate and the multivariate PSDs, but the stress was on the inference side, see *http://www.stat.psu.edu/~gpp/contribu.htm* for the detailed contributions. In his 1966 paper, Patil discussed in detail the multivariate PSDs, and in particular made the remark: "On the lines of Khatri (1959), one has the result that the multivariate generalized PSD is uniquely determined by its vector of means and the variance-covariance matrix given as functions of the parameters θ_1, θ_2, ..., θ_k ". Later Al Jasim (1975), proved that a PSD is uniquely determined by any one of its cumulant. Finally, the rule and importance of the PSDs in stochastics modeling and Bayesian inference, were studied by Abdulrazak and Patil in 1986.

Some work on characterizations of the univariate PSDs was considered by Abdulrazak and Patil. In particular, in their 1994 paper they gave characterizations of the univariate PSDs via different stochastic characteristics models, namely, the urn, the mixture, the monotone likelihood ratio, the birth, the Hermite, the renewal, and the additive damage models. While in their 1996 paper they gave characterizations to some PSDs using form-invariance property of weighted sampling

SOME CHARACTERIZATION THEOREMS FOR THE BIVARIATE POWER SERIES DISTRIBUTIONS

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Abstruct

Discrete distributions having the property that their probability mass functions are proportional to the terms of certain power series functions, are called power series distributions. In this paper, after defining the univariate and the bivariate power series distributions (BPSDs), we gave some examples of well-known bivariate discrete distributions that are BPSDs, as well as some new ones. Then, we studied the marginal distributions and the bivariate moments of the BPSDs. We obtained many interesting recurrence relations for the moments and the bivariate moments. These relations are used to uniquely determine the BPSDs. These results are also used to construct more examples of the BPSDs.

1. Introduction

The power series distributions (PSDs) are a family of distributions having a certain functional form, and play an important rule in statistics and its applications. The PSDs were defined for the first time in 1949 by